The Standard Model

Let Ω be the set of what can happen in the world, be the possible trading times, a collection of increasingly fine partitions of Ω representing the information available at each time, and a positive measure with mass 1 on the algebra generated by the partitions.

The *standard model* specifies *prices* , and *cash flows* , where is the set of available market instruments. Instrument prices are assumed to be perfectly liquid: they can be bought and sold at the same price in any amount. Cash flows are associated with owning an instrument: stocks have dividends, bonds have coupons, futures have margin adjustments.

A *trading strategy* is a finite collection of strictly increasing stopping times, , and trades, indicating the number of shares to trade in each instrument. Trades accumulate to a *position*, where when .

The *value* of a position at time is : also called *marked-to-market*, is how much you would get from liquidating your position and the trades just executed assuming you could do that. The *amount* generated by the trading strategy at time is : you receive the cash flows associated with your existing position and pay for the trades you just executed.

A process is a martingale if . This is defined by where . If is understood we write this as . The usual notation is or more succinctly .

A model has *strict arbitrage* if there is no trading strategy with and for . The Fundamental Theorem of Asset Pricing states there is not strict arbitrage if and only if there exists a positive adapted process, , with

Note that if for all this says is a martingale.

A simple corollary using the definition of value and amount shows

For a trading strategy with , and for , . Since , , and we have , where the 0 subscript denote time .

Every model of the form where is a martingale and is a positive adapted process is strict arbitrage-free. This is immediate by substituting in equation (1).

# Derivative Securities

Equation (2) is the skeleton key to understanding derivative securities. A *derivative* is a contract between a buyer and a seller for exchanging instruments over time. Assuming the instruments can be cash settled, a contract specifies a sequence of times, , and cash amounts, , to be given to the buyer by the seller. If there is a trading strategy that replicates the contract, then its value at time is . Since we have the Fréchet derivative of is.

Define the stopping time then